

FRAME CONSTANTS OF GABOR FRAMES NEAR THE CRITICAL DENSITY

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ABSTRACT. We consider Gabor frames generated by a Gaussian function and describe the behavior of the frame constants as the density of the lattice approaches the critical value.

1. INTRODUCTION

In this article we study the stability problem for the expansions of functions on the real line with respect to a discrete set of phase-space shifts of a Gaussian, precisely

$$f(x) = \sum_{k,l \in \mathbb{Z}} c_{kl} e^{2\pi i l a x} e^{-\pi(x-bk)^2}. \quad (1.1)$$

Expansions of such form (with $a = 1$, $b = 1$) were introduced by D. Gabor in his classical article [5]. Now expansions of type (1.1), so-called Gabor expansions, appear in signal processing, quantum mechanics, time-frequency analysis, the theory of pseudodifferential operators, and other applications.

During the last decades an extensive theory of expansions (1.1) as well as more general Gabor expansions has been developed (see, for instance [3, 6] and the references therein). However, not much is known about numerical stability property of such expansions.

In modern language, the existence of Gabor expansions is derived from frame theory. To fix terminology and notation, take some $g \in L^2(\mathbb{R})$, it will be called a window function, and let $\Lambda = M\mathbb{Z}^2 \subset \mathbb{R}^2$ be a lattice in \mathbb{R}^2 , where M is a 2×2 invertible real-valued matrix. Given a point $\lambda = (x, \xi)$ in phase-space \mathbb{R}^2 , the corresponding time-frequency shift is

$$\pi_\lambda f(t) = e^{2\pi i \xi t} f(t - x), \quad t \in \mathbb{R}.$$

The set of functions $\mathcal{G}(g, \Lambda) = \{\pi_\lambda g : \lambda \in \Lambda\}$ is called the *Gabor system* generated by g and Λ . We say that such a system is a *Gabor frame* or *Weyl-Heisenberg frame*, whenever there exist constants $A, B > 0$ such that, for all $f \in L^2(\mathbb{R})$,

$$A\|f\|_{L^2(\mathbb{R})}^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi_\lambda g \rangle_{L^2(\mathbb{R})}|^2 \leq B\|f\|_{L^2(\mathbb{R})}^2. \quad (1.2)$$

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The (best possible) constants $A = A(\Lambda, g)$ and $B = B(\Lambda, g)$ in (1.2) are called the *lower and upper frame bounds* for the frame $\mathcal{G}(g, \Lambda)$.

There is a standard procedure for constructing expansions of type (1.1) for each Gabor frame. Namely there exists a *dual window* $\gamma \in L^2(\mathbb{R})$, such that every $f \in L^2(\mathbb{R})$ can be expanded as a Gabor series

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi_\lambda \gamma \rangle \pi_\lambda g. \quad (1.3)$$

Such dual window is, in general, non-unique. We refer the reader to e.g. [6] for an exposition of Gabor analysis and related matters.

The property of the system $\mathcal{G}(g, \Lambda)$ to form a frame in $L^2(\mathbb{R})$ depends (among other factors) on geometrical characteristics of Λ . We say that the area of its fundamental domain $s(\Lambda) = \text{Area}(M[0, 1]^2) = |\det M|$ is the *size* of Λ . By the *density* of Λ we mean $d(\Lambda) = s(\Lambda)^{-1}$; for the lattice case this definition of density coincides with numerous standard density definitions (see e.g. [16]).

We refer to [9] for a comprehensive account of the density theorems for Gabor frames. In fact for *any* window function g the condition $s(\Lambda) \leq 1$ is necessary for $\mathcal{G}(g, \Lambda)$ to be a frame in $L^2(\mathbb{R})$. For “nice” windows g (in the Schwartz class, say) a fascinating form of the uncertainty principle, the so-called Balian-Low Theorem (BLT), requires even that $s(\Lambda) < 1$ for $\mathcal{G}(g, \Lambda)$ to be a frame [2].

The results of [13, 17] yield in particular that in the case of the Gaussian window

$$g_0(t) = e^{-\pi t^2}, \quad t \in \mathbb{R}.$$

the condition $s(\Lambda) < 1$ is also sufficient:

Theorem A. *The set $\mathcal{G}(g_0, \Lambda)$ is a frame for $L^2(\mathbb{R})$ if and only if $s(\Lambda) < 1$.*

Together with BLT this implies that the lower frame bound $A = A(\Lambda)$ must tend to 0, as the size of the lattice $s(\Lambda)$ approaches one. Thus the original Gabor series (1.1) with $a = 1$, $b = 1$ corresponds to the critical case $s(\Lambda) = 1$ and does not provide an L^2 -stable expansion.¹ In this case, there exist L^2 -functions with polynomially growing coefficients. See [10, 15] for the convergence properties of (1.1).

In this article we are concerned exclusively with Gabor frames for the Gaussian window $\mathcal{G}(g_0, \Lambda)$ for the square lattice $\Lambda(a) = a\mathbb{Z} \times a\mathbb{Z}$ and study the behavior of its frame constants $A(a) = A(\Lambda(a))$ and $B(a) = B(\Lambda(a))$ near the critical density $d(\Lambda) = 1$. The main result of the article reads as follows.

Theorem 1.1. *There exist constants $0 < c < C < \infty$ such that for each $a \in (1/2, 1)$ the frame bounds $A(a)$, $B(a)$ for the frame $\mathcal{G}(g_0, \Lambda(a))$ satisfy*

$$c(1 - a^2) \leq A(a) \leq C(1 - a^2) \quad (1.4)$$

and

$$c < B(a) < C. \quad (1.5)$$

¹In his article [5] Gabor considered expansions of functions f which possess additional decay in time and frequency. As we now know (see e.g. [15]), for such functions the series (1.1) converges in $L^2(\mathbb{R})$.

Remark 1.2. A similar statement holds for arbitrary rectangular lattices. The values of c, C in this theorem then depend upon the shape of the lattice. Nevertheless, one can prove that given a number $K > 0$ there exist constants c and C valid for all matrices M such that the diameter of the fundamental domain $M(0, 1]^2$ does not exceed K .

The ratio $B(\Lambda)/A(\Lambda)$ plays the role of the condition number for the frame $\mathcal{G}(g_0, \Lambda)$. Thus Theorem 1.1 says how fast does the frame $\mathcal{G}(g_0, \Lambda)$ "numerically degenerate" as its density approaches the critical value.

The asymptotical behavior $A(a) \asymp (1 - a^2)$ has been first observed numerically by Thomas Strohmer [19] and by Peter Sondergaard [18]. Moreover, the numerical simulation in [19] allowed us to guess the construction which gives the second inequality in (1.4). This construction is described in Section 4 below.

Next let $g_1(t) = (\cosh \pi \gamma t)^{-1}$, $\gamma > 0$, be the hyperbolic cosine function. Janssen and Strohmer [12] have shown that $\mathcal{G}(g_1, a\mathbb{Z} \times b\mathbb{Z})$ is a frame, if and only if $ab < 1$. To do this, they showed that the frame bounds for $\mathcal{G}(g_1, a\mathbb{Z} \times b\mathbb{Z})$ are equivalent to those of $\mathcal{G}(g_0, a\mathbb{Z} \times b\mathbb{Z})$ with the Gaussian g_0 and applied Theorem A. Therefore we obtain the same asymptotic estimates for the frame bounds for the hyperbolic cosine.

Corollary 1.3. *There exist constants $0 < c < C < \infty$ such that for each $a \in (1/2, 1)$ the frame bounds $\tilde{A}(a)$, $\tilde{B}(a)$ for the frame $\mathcal{G}(g_1, \Lambda(a))$ satisfy*

$$c(1 - a^2) \leq \tilde{A}(a) \leq C(1 - a^2)$$

and

$$c \leq \tilde{B}(a) \leq C.$$

The proof of Theorem 1.1 involves both time-frequency methods and methods of complex analysis. We use complex analysis in order to obtain the upper estimate for $A(a)$ and the Gabor analysis in order to obtain the rest of the statements in Theorem 1.1 (though a pure complex-analytic proof is also available). In particular we apply Walnut's estimates for the norm of the frame operator [20], and also precise decay estimates for the dual window established in [7]. The upper bound $A(a) \leq C(1 - a^2)$ will be established by the construction of a concrete example. We produce a function f_a (depending on the lattice $\Lambda(a)$), such that

$$\sum_{\lambda \in \Lambda(a)} |\langle f_a, \pi_\lambda g_0 \rangle|^2 \leq C(1 - a^2) \|f_a\|_{L^2(\mathbb{R})}^2.$$

By using the Bargmann transform, we translate our problem into one of finding entire functions in the Bargmann-Fock space whose restrictions to $\Lambda(a)$ are "small" with respect to their Fock norms.

The paper is organized as follows. In the next section we discuss the estimates for $B(a)$. Furthermore, we give the lower estimates for $A(a)$. Here we mainly follow the arguments from [7]. In section 3 we recall the definition of the Fock space \mathcal{F} of entire functions, and discuss the relations between the frame property of the system

$\mathcal{G}(g_0, \Lambda(a))$ and sampling in \mathcal{F} . We also recall basic properties of the Weierstrass σ -function. In section 4 we use these facts and also special "atomization" techniques in order to construct the example which delivers the upper estimate in (1.4).

Notation: To avoid dealing with too many intermediate constants, we use the standard notation $f \prec g$ to express an inequality $f(x) \leq Cg(x)$ for all x with a constant C independent of x (and possibly other parameters). Likewise, $f \asymp g$ means that there exist $A, B > 0$ such that $Af(x) \leq g(x) \leq Bf(x)$ for all x .

2. TIME-FREQUENCY METHODS TO ESTIMATE FRAME BOUNDS

The estimate (1.5) on the upper frame bound $B(a)$ can be obtained in various ways. In particular, we can use Walnut's estimates, which give a sufficient condition for the Gabor frame operator to be bounded [20]. This result also follows from the Polya-Plancherel type inequalities for functions in the Bargmann-Fock space, see below Section 3 for more details.

To obtain the lower estimates for $A(a)$ we need to show the invertibility of the Gabor frame operator and to estimate the norm of the inverse operator. We will approach this problem by using information about a suitable dual window γ and then apply Walnut's estimates to the Gabor expansion (1.3).

To state Walnut's result we need the following definitions.

Let W be the Wiener amalgam space of functions on the real line defined by the norm

$$\|g\|_W = \sum_{k \in \mathbb{Z}} \sup_{t \in [0,1]} |g(t+k)|.$$

Given a function g in $L^2(\mathbb{R})$, consider the Gabor system $\mathcal{G}(g, \Lambda)$ and the corresponding synthesis operator $D_{g,\Lambda}$,

$$D_{g,\Lambda} \mathbf{c} = \sum_{\lambda \in \Lambda} c_\lambda \pi_\lambda g,$$

and the analysis operator $C_{g,\Lambda}$,

$$(C_{g,\Lambda} f)(\lambda) = \langle f, \pi_\lambda g \rangle, \quad \lambda \in \Lambda.$$

If $D_{g,\Lambda}$ acts continuously from $\ell^2(\Lambda)$ to $L^2(\mathbb{R})$, then $C_{g,\Lambda}$ acts continuously from $L^2(\mathbb{R})$ to $\ell^2(\Lambda)$, and $C_{g,\Lambda} = D_{g,\Lambda}^*$.

The following lemma from [20] gives an estimate for $\|D_{g,\Lambda}\|_{\ell^2 \rightarrow L^2}$:

Lemma 2.1. *If $g \in W$ and $\Lambda = a\mathbb{Z}^2$, then $D_{g,\Lambda}$ is bounded from $\ell^2(\Lambda)$ to $L^2(\mathbb{R})$ and*

$$\|D_{g,\Lambda}\|_{L^2(\mathbb{R})} \leq (1 + a^{-1}) \|g\|_W.$$

Since, obviously, in our situation $B(a) = \|C_{g,\Lambda}\|_{L^2 \rightarrow \ell^2}^2 = \|D_{g,\Lambda}\|_{\ell^2 \rightarrow L^2}^2$, we obtain

Corollary 2.2. *If $g \in W$ and $a > 0$, then*

$$B(a) \leq (1 + a^{-1})^2 \|g\|_W^2. \quad (2.1)$$

To treat Gabor frames with Gaussian window, we need to evaluate the amalgam space norm of functions with Gaussian decay.

Lemma 2.3. *Assume that $\kappa > 0$, $|\gamma(t)| \leq e^{-\pi\kappa t^2}$. Then*

$$\|\gamma\|_W \leq 2 + \kappa^{-1/2}. \quad (2.2)$$

Proof. For $n \geq 1$, $n \in \mathbb{Z}$, we have

$$\sup_{t \in [0,1]} |\gamma(n+t)| \leq e^{-\pi\kappa n^2} \leq \int_{n-1}^n e^{-\pi\kappa t^2} dt,$$

and likewise for $n < -1$, $n \in \mathbb{Z}$, we have

$$\sup_{t \in [0,1]} |\gamma(n+t)| \leq e^{-\pi\kappa(|n|-1)^2} \leq \int_{|n|-2}^{|n|-1} e^{-\pi\kappa t^2} dt.$$

Consequently,

$$\|\gamma\|_W = \sum_{n \in \mathbb{Z}} \sup_{t \in [0,1]} |\gamma(t+n)| \leq 2 + \int_{\mathbb{R}} e^{-\pi\kappa t^2} dt = 2 + \kappa^{-1/2}.$$

□

As a consequence we obtain an estimate on the upper frame bound of Gaussian Gabor frames.

Proposition 2.4. *The upper frame bound $B(a)$ of $\mathcal{G}(g_0, a\mathbb{Z}^2)$, $1/2 < a < 1$, satisfies the estimate*

$$1 < B(a) < 100.$$

Proof. For the upper estimate we use (2.1) and (2.2) with $\kappa = 1$.

To get the lower estimate we consider the sum (1.2) for $f = g = g_0$. Then

$$\sum_{\lambda \in \Lambda(a)} |\langle g_0, \pi_\lambda g_0 \rangle|^2 > \|g_0\|^2,$$

which yields the desired estimate. □

The time-frequency methods also yield the lower estimate in (1.4). This estimate requires the existence and some knowledge about a dual window. If $\mathcal{G}(g, \Lambda)$ is a frame, then by the frame theory there exists a dual window $\gamma \in L^2(\mathbb{R})$, such that every $f \in L^2(\mathbb{R})$ possesses a(n unconditionally convergent) series expansion (Gabor expansion) of the form

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi_\lambda g \rangle \pi_\lambda \gamma = D_{\gamma, \Lambda} C_{g, \Lambda} f.$$

For the square lattice $\Lambda(a)$, Lemma 2.1 yields the following bound:

$$\|f\|_{L^2(\mathbb{R})}^2 \leq (a^{-1} + 1)^2 \|\gamma\|_W^2 \sum_{\lambda \in \Lambda} |\langle f, \pi_\lambda g \rangle|^2.$$

Consequently, the lower frame bound $A(a)$ can be estimated from below as

$$A(a) \geq ((a^{-1} + 1)^2 \|\gamma\|_W^2)^{-1}. \quad (2.3)$$

Proposition 2.5. *For the square lattice $\Lambda(a)$, $1/2 < a < 1$, the lower bound $A(a)$ of the Gaussian frame $\mathcal{G}(g_0, \Lambda(a))$ obeys the estimate*

$$A(a) \succ 1 - a^2.$$

Proof. In [7] the authors consider the Gaussian Gabor frame $\mathcal{G}(g_0, \Lambda(a))$. For this frame they construct a dual window γ such that

$$|\gamma(t)| \leq Ce^{-\pi\kappa t^2}$$

with $\kappa \asymp 1 - a^2$.

By Lemma 2.3 we have

$$\|\gamma\|_W \prec 2 + \kappa^{-1/2} \prec (1 - a^2)^{-1/2},$$

and the desired estimate follows now from (2.3). \square

3. COMPLEX METHODS

3.1. Fock space. We recall the definition and basic properties of the Fock space. We refer the reader to [4], [6] for detailed proofs and also for a discussion of numerous applications of this space to signal analysis and quantum mechanics.

The *Fock space* \mathcal{F} is the Hilbert space of all entire functions such that

$$\|F\|_{\mathcal{F}}^2 := \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm_z < \infty,$$

where dm_z is Lebesgue measure on \mathbb{C} .

The natural inner product in \mathcal{F} is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{F}}$.

We will use the following well-known facts:

(a) The point evaluation is a bounded linear functional in \mathcal{F} , and the corresponding reproducing kernel is the function $w \mapsto e^{\pi\bar{z}w}$, i.e.,

$$F(z) = \langle F, e^{\pi\bar{z}\cdot} \rangle_{\mathcal{F}}, \quad F \in \mathcal{F}. \quad (3.1)$$

(b) One defines the Bargmann transform of a function $f \in L^2(\mathbb{R})$ by

$$f \mapsto \mathcal{B}f(z) = F(z) = 2^{1/4} e^{-\pi z^2/2} \int_{\mathbb{R}} f(t) e^{-\pi t^2} e^{2\pi tz} dt.$$

The Bargmann transform is a unitary mapping from $L^2(\mathbb{R})$ onto \mathcal{F} .

(c) In what follows we identify \mathbb{C} and \mathbb{R}^2 . In particular for each $\zeta = \xi + i\eta \in \mathbb{C}$ we write $\pi_{\zeta} = \pi_{(\xi, \eta)}$. Define the Fock space shift $\beta_{\zeta} : \mathcal{F} \rightarrow \mathcal{F}$ by

$$\beta_{\zeta} F(z) = e^{i\pi\xi\eta} e^{-\pi|\zeta|^2/2} e^{\pi\zeta z} F(z - \bar{\zeta}).$$

Then β_{ζ} is unitary on \mathcal{F} , and the Bargmann transform intertwines the Fock space shift and the time-frequency shift:

$$\beta_{\zeta} \mathcal{B} = \mathcal{B} \pi_{\zeta}. \quad (3.2)$$

(d)

$$\mathcal{B}g_0 = 2^{-1/4}, \quad (3.3)$$

here as above g_0 is the Gaussian function.

(e) It follows from (3.2) and (3.3) that

$$\mathcal{B}\pi_\zeta g_0 = 2^{-1/4} e^{i\pi\xi\eta} e^{-\pi|\zeta|^2/2} e^{\pi\zeta z}.$$

Taking into account the reproducing property (3.1), we can rewrite the frame property (1.2) of $\mathcal{G}(g_0, \Lambda)$ as the sampling inequality

$$A\|F\|_{\mathcal{F}}^2 \leq \frac{1}{2} \sum_{\lambda \in \Lambda} |F(\bar{\lambda})|^2 e^{-\pi|\lambda|^2} \leq B\|F\|_{\mathcal{F}}^2, \quad F \in \mathcal{F}.$$

In the case of square lattice, Λ is symmetric with respect to the real line, and we have

$$A\|F\|_{\mathcal{F}}^2 \leq \frac{1}{2} \sum_{\lambda \in \Lambda} |F(\lambda)|^2 e^{-\pi|\lambda|^2} \leq B\|F\|_{\mathcal{F}}^2, \quad F \in \mathcal{F}. \quad (3.4)$$

(f) Let $1/2 < a < 2$, and let $w \in \mathbb{C}$, $w \neq 0$. Consider the entire function $\Phi_{a,w}(z) = e^{a\bar{w}z^2/w}$. Then

$$|\Phi_{a,w}(z)| \asymp e^{a|z|^2}, \quad |z - w| < 1.$$

This statement can be checked by direct inspection.

3.2. Reformulation of the main result. The remaining part of Theorem 1.1 can now be reformulated as follows.

Theorem 3.1. *Let $\Lambda(a) = a\mathbb{Z}^2$, and let $A(a)$ be the best possible constant in the left hand side inequality of (3.4). Then for $1/2 < a < 1$ we have*

$$A(a) \prec 1 - a^2.$$

To prove this theorem we need to find a constant K and functions $F = F_a \in \mathcal{F}$ such that

$$K(1 - a^2)\|F_a\|^2 \geq \sum_{\lambda \in \Lambda(a)} |F_a(\lambda)|^2 e^{-\pi|\lambda|^2}. \quad (3.5)$$

3.3. Weierstrass σ -function. The construction of the functions F_a in the next section is motivated by the properties of the classical Weierstrass σ -function. Let us recall its definition and basic properties. We refer the reader to [1] for a systematic study of this function and also to [7] for its applications in Gabor analysis.

Given a lattice $\Lambda \subset \mathbb{C}$ we denote

$$\sigma(\Lambda, z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) e^{\frac{z}{\lambda} + \frac{1}{2}\left(\frac{z}{\lambda}\right)^2}.$$

This product converges uniformly on compact sets in \mathbb{C} to an entire function with Λ as the zero set. This is a function of order 2; moreover there exists $d_\Lambda \in \mathbb{C}$ such that

$$|\sigma(\Lambda, z) e^{d_\Lambda z^2}| \asymp e^{\frac{\pi}{2}s(\Lambda)^{-1}|z|^2}, \quad \text{dist}(\Lambda, z) \geq \varepsilon > 0.$$

Here $s(\Lambda)$ is the area of the fundamental domain of Λ . See [8] and also [7].

Once again, let $\Lambda(a) = a\mathbb{Z}^2$. A direct inspection shows that $d_{\Lambda(a)} = 0$, so that

$$|\sigma(\Lambda(a), z)| \asymp e^{\frac{\pi}{2}a^{-2}|z|^2}, \quad \text{dist}(\Lambda_a, z) \geq \varepsilon > 0. \quad (3.6)$$

This relation allows one to mimic the weight function $e^{\pi|z|^2/2}$ in the definition of the Fock space by the absolute value of an analytic function.

4. PROOF OF (3.5)

4.1. Explicit construction. If a is in a compact subinterval of $(1/2, 1)$, one can take $F_a = 1$ and obtain (3.5) with some appropriate constant K . Therefore, from now on, we assume that a is sufficiently close to 1, say $0.999 < a < 1$. Given such a , we take $R = R(a)$ such that

$$2(1 - a^2) < R^{-3/2} < 4(1 - a^2)$$

and

$$n_R := \pi(1 - R^{-3/2})R^2 \in \mathbb{N}.$$

We need some additional notation:

$$\begin{aligned} b^2 &= 1 - R^{-3/2}, \\ \zeta_{m,n} &= b^{-1}(m + in), \\ Q_{m,n} &= \{x + iy \in \mathbb{C} : |x - b^{-1}m| < b^{-1}/2, |y - b^{-1}n| < b^{-1}/2\}, \\ D_R &= \{z \in \mathbb{C} : |z| < R\}, \\ D'_R &= \cup\{Q_{m,n} : |\zeta_{m,n}| < R - 3\}, \quad D''_R = D_R \setminus D'_R, \\ \mathcal{N}_R &= \{(m, n) \in \mathbb{Z}^2 : Q_{m,n} \subset D'_R\}, \\ q_R &= \text{Card } \mathcal{N}_R, \quad p_R = n_R - q_R. \end{aligned}$$

We have

$$\{z : R - 1 < |z| < R\} \subset D''_R \subset \{z : R - 4 < |z| < R\}.$$

Using the appropriate segments of radii of the disc D_R we split D''_R into the "sectors" A_k :

$$D''_R = \bigcup_{k=1}^{p_R} A_k,$$

such that

$$m < \text{diam } A_k < M, \quad \text{Area } A_k = b^{-2}$$

for some m, M independent of a . Denote the center of mass of A_k by

$$\zeta_k = b^2 \int_{A_k} \zeta dm_\zeta. \quad (4.1)$$

We can find c independent of a such that

$$\{w : |w - \zeta_k| < c\} \subset A_k, \quad k = 1, \dots, p_R.$$

We are going to verify the estimate (3.5) for the function

$$F_a(z) = z \prod_{(m,n) \in \mathcal{N}_R \setminus (0,0)} \left(1 - \frac{z}{\zeta_{m,n}}\right) \prod_{k=1}^{p_R} \left(1 - \frac{z}{\zeta_k}\right). \quad (4.2)$$

The zero set of the function F_a is

$$\mathcal{Z}_a = \{\zeta_{m,n} : |\zeta_{m,n}| < R - 3\} \cup \{\zeta_k\}_{k=1}^{p_R}. \quad (4.3)$$

By construction, the total number of zeros of F_a is $n_R = \pi R^2 b^2$.

In order to prove (3.5), we need to estimate both $\|F_a\|_{\mathcal{F}}^2$ and

$$\|F_a\|_a^2 := \sum_{m,n \in \mathbb{Z}} |F_a(a(m + in))|^2 e^{-\pi a^2(m^2 + n^2)}.$$

4.2. Estimate of $\|F_a\|_{\mathcal{F}}^2$. To estimate the norm of F_a in the Fock space, we compare the logarithm of the modulus of the polynomial F_a to a subharmonic function u_R whose growth is easy to control.

Consider the subharmonic function

$$u_R(z) = \int_{|\zeta| < R} \log \left| 1 - \frac{z}{\zeta} \right| dm_\zeta = \begin{cases} \frac{\pi}{2} |z|^2, & |z| < R, \\ \pi R^2 \log |z| - \pi R^2 \log R + \frac{\pi}{2} R^2, & |z| > R. \end{cases}$$

An easy estimate shows that

$$u_R(z) < \frac{\pi}{2} |z|^2, \quad |z| > R.$$

We use the following approximation lemma.

Lemma 4.1. *For each $\varepsilon > 0$ there exist constants $0 < c(\varepsilon) < C(\varepsilon) < \infty$ such that*

$$c(\varepsilon) |F_a(z)| < e^{b^2 u_R(z)} < C(\varepsilon) |F_a(z)|, \quad \text{dist}(z, \mathcal{Z}_a) > \varepsilon. \quad (4.4)$$

and

$$c(\varepsilon) |F_a(z)| < e^{b^2 u_R(z)}, \quad \text{dist}(z, \mathcal{Z}_a) \leq \varepsilon. \quad (4.5)$$

Remark. Since the set \mathcal{N} is invariant with respect to rotation by $\pi/2$ around the origin, we find that

$$\sum_{(m,n) \in \mathcal{N}_R \setminus (0,0)} \frac{1}{m + in} = \sum_{(m,n) \in \mathcal{N}_R \setminus (0,0)} \frac{1}{(m + in)^2} = 0.$$

So the first factor on the right-hand side of (4.2) in the definition of F_a can be written as

$$\begin{aligned} V_R(z) &= z \prod_{(m,n) \in \mathcal{N}_R \setminus (0,0)} \left(1 - \frac{z}{\zeta_{m,n}} \right) \\ &= z \prod_{(m,n) \in \mathcal{N}_R \setminus (0,0)} \left(1 - \frac{z}{\zeta_{m,n}} \right) \exp \left(\frac{z}{\zeta_{m,n}} + \frac{1}{2} \left(\frac{z}{\zeta_{m,n}} \right)^2 \right). \end{aligned} \quad (4.6)$$

Consequently, the function V_R can be viewed as a truncated version of the Weierstrass σ -function and estimates (4.4) and (4.5) correspond to the growth estimate (3.6) for the Weierstrass σ -function.

We postpone the proof of this technical lemma until subsection 4.4. Assuming that Lemma 4.1 is already proved, an estimate of $\|F_a\|_{\mathcal{F}}^2$ is straightforward.

Lemma 4.2.

$$\|F_a\|_{\mathcal{F}}^2 \asymp R^{3/2} \asymp (1 - a^2)^{-1}.$$

Proof. Let $\Omega = \{z \in \mathbb{C} : |z| < R, \text{dist}(z, \mathcal{Z}_a) > 1/10\}$. Using Lemma 4.1, we find that

$$\|F_a\|_{\mathcal{F}}^2 \succ \int_{\Omega} e^{2b^2 u_R(z) - \pi|z|^2} dm_z = I(a, R).$$

We use that $1 - b^2 = R^{-3/2}$. Furthermore, for every $1 < r < R$, the circle $z : |z| = r$ intersects with Ω on at most half of its length. Therefore,

$$I(a, R) \succ \int_1^R e^{-\pi R^{-3/2} t^2} t dt = \frac{R^{3/2}}{2} \int_{R^{-3/2}}^{R^{1/2}} e^{-\pi u} du \asymp R^{3/2},$$

and the statement of the Lemma now follows. \square

Remark. A similar argument shows that

$$\int_{|z| > R-4} |F_a(z)|^2 e^{-\pi|z|^2} dm_z \rightarrow 0, \quad (4.7)$$

as $a \rightarrow 1$ or equivalently, as $R \rightarrow \infty$.

4.3. Estimate of $\|F_a\|_a^2$.

Lemma 4.3. For F_a as in (4.2) we have

$$\|F_a\|_a^2 = \sum_{m,n} |F_a(a(m+in))|^2 e^{-\pi a^2(m^2+n^2)} \asymp 1. \quad (4.8)$$

Proof. We have

$$\|F_a\|_a^2 = \left(\sum_{(m,n) \in \mathcal{N}_R} + \sum_{(m,n) \notin \mathcal{N}_R} \right) |F_a(a(m+in))|^2 e^{-\pi a^2(m^2+n^2)} = \Sigma_1(a) + \Sigma_2(a).$$

In order to estimate $\Sigma_1(a)$, we observe that $F_a(\zeta_{m,n}) = 0$, for $(m,n) \in \mathcal{N}_R$ and

$$|\zeta_{m,n} - a(m+in)| = (b^{-1} - a)|m+in| < 2R^{-3/2}|m+in|.$$

Since $|m+in| < R$ for $(m,n) \in \mathcal{N}_R$, and a is sufficiently close to 1, we have

$$|\zeta_{m,n} - a(m+in)| < \frac{1}{5}, \quad (m,n) \in \mathcal{N}_R.$$

Denote $D_{m,n} = \{z \in \mathbb{C} : |z - \zeta_{m,n}| < 1/4\}$. By part (f) of subsection 3.1 there exists a function $\Phi_{m,n}(z)$ that is holomorphic on $D_{m,n}$ and satisfies

$$|\Phi_{m,n}(z)| \asymp e^{b^2 \frac{\pi}{2} |z|^2}, \quad z \in D_{m,n}. \quad (4.9)$$

Then for each $(m,n) \in \mathcal{N}_R$ the function

$$\Psi_{m,n}(z) = \frac{F_a(z)}{\Phi_{m,n}(z)}$$

is holomorphic in $D_{m,n}$ and possesses the properties

$$\Psi_{m,n}(\zeta_{m,n}) = 0 \text{ and } |\Psi_{m,n}(z)| \prec 1.$$

By Cauchy's theorem, the functions $\Psi'_{m,n}$ are uniformly bounded on $D_{m,n}^* = \{z \in \mathbb{C} : |z - \zeta_{m,n}| < 1/5\}$, and hence

$$|\Psi_{m,n}(a(m+in))| \prec |(a - b^{-1})(m+in)| \prec R^{-3/2}(m^2 + n^2)^{1/2}, \quad (m,n) \in \mathcal{N}_R.$$

Returning to the function F_a and using (4.9) once again, we obtain

$$|F_a(a(m + in))|^2 e^{-\pi a^2(m^2+n^2)} \prec R^{-3}(m^2 + n^2) |\Phi_{m,n}(a(m + in))|^{2(1-b^{-2})}.$$

The mean value inequality for $|\Phi_{m,n}(a(m + in))|^{2(1-b^{-2})}$ now yields

$$\begin{aligned} |F_a(a(m + in))|^2 e^{-\pi a^2(m^2+n^2)} &\prec R^{-3}(m^2 + n^2) \int_{D_{m,n}} |\Phi_{m,n}(z)|^{2(1-b^{-2})} dm_z \\ &\prec R^{-3} \int_{D_{n,m}} |z^2| e^{\pi(b^2-1)|z|^2} dm_z. \end{aligned}$$

Since all discs $D_{m,n}$ are disjoint we obtain

$$\Sigma_1(a) \prec R^{-3} \int_{|z|<R} |z^2| e^{\pi(b^2-1)|z|^2} dm_z \prec R^{-3} \int_0^R t^2 e^{-\pi R^{-3/2}t^2} t dt \prec 1. \quad (4.10)$$

Finally, for arbitrary (m, n) , the mean value theorem yields

$$|F_a(a(m + in))|^2 e^{-\pi a^2(m^2+n^2)} \prec \int_{D_{m,n}} |F_a(z)|^2 e^{-\pi|z|^2} dm_z,$$

and, by (4.7) we obtain

$$\Sigma_2(a) \prec \int_{|z|>R-4} |F_a(z)|^2 e^{-\pi|z|^2} dm_z \rightarrow 0, \text{ as } a \nearrow 1. \quad (4.11)$$

The estimates (4.10) and (4.11) yield

$$\|F_a\|_a^2 = \sum_{m,n} |F_a(a(m + in))|^2 e^{-\pi a^2(m^2+n^2)} \prec 1. \quad (4.12)$$

The opposite relation follows from Lemma 4.2 and from the lower estimate on $A(a)$ established in Proposition 2.5. \square

Relation (3.5) follows immediately from Lemma 4.2 and the estimate (4.8) (or even (4.12)).

4.4. Proof of the approximation lemma. The proof of Lemma 4.1 is based on atomization techniques, see e.g. [14]. First we rewrite (4.4) and (4.5) in an additive form. We must prove that

$$\log |F_a(z)| = b^2 u_R(z) + O(1), \quad \text{dist}(z, \mathcal{Z}_a) > \varepsilon, \quad (4.13)$$

and

$$\log |F_a(z)| \leq b^2 u_R(z) + O(1), \quad \text{dist}(z, \mathcal{Z}_a) \leq \varepsilon,$$

where $\mathcal{Z}_a = \{\zeta_{m,n} : (m, n) \in \mathcal{N}_R\} \cup \{\zeta_k\}_{k=1}^{p_R}$ as in (4.3), and the quantities $O(1)$ in the right-hand sides of these relations are bounded uniformly with respect to all $a \in (0.999, 1)$ and depend only on ε . It suffices to prove (4.13), the second relation will then follow by the maximum principle applied to $F_a \Phi_{a,w}^{-1}$ where $\Phi_{a,w}$ is defined in part (f) of subsection 3.1.

Let V_R be defined by (4.6),

$$v_R(z) = b^2 \int_{D'_R} \log \left| 1 - \frac{z}{\zeta} \right| dm_\zeta,$$

and let

$$w_R(z) = b^2 \int_{D''_R} \log \left| 1 - \frac{z}{\zeta} \right| dm_\zeta, \quad W_R(z) = \prod_1^{p_R} \left(1 - \frac{z}{\zeta_k} \right).$$

We have

$$\begin{aligned} \log |F_a(z)| - b^2 u_R(z) = \\ (\log |V_R(z)| - v_R(z)) + (\log |W_R(z)| - w_R(z)) = \mathfrak{S}_1(R, z) + \mathfrak{S}_2(R, z), \end{aligned} \quad (4.14)$$

and we estimate separately each summand in the right-hand side of (4.14).

Let $\text{dist}(z, \mathcal{Z}_a) > \varepsilon$. We have

$$\begin{aligned} \mathfrak{S}_1(R, z) = \log |V_R(z)| - v_R(z) = b^2 \int_{Q_{0,0}} \left(\log |z| - \log \left| 1 - \frac{z}{\zeta} \right| \right) dm_\zeta \\ + b^2 \left(\sum_{\substack{(m,n) \in \mathcal{N}_R \setminus \{0,0\}, \\ \text{dist}(z, Q_{m,n}) \leq 10}} + \sum_{\substack{(m,n) \in \mathcal{N}_R \setminus \{0,0\}, \\ \text{dist}(z, Q_{m,n}) > 10}} \right) \underbrace{\int_{Q_{m,n}} \left(\log \left| 1 - \frac{z}{\zeta_{m,n}} \right| - \log \left| 1 - \frac{z}{\zeta} \right| \right) dm_\zeta}_{j_{m,n}}. \end{aligned}$$

It suffices to estimate just the second sum in the right-hand side because the first sum contains only a finite number (at most 1000, say) of uniformly bounded terms, and the first integral is bounded uniformly in z .

Denote $L(\zeta) = \log(1 - z/\zeta)$. We then have

$$j_{m,n} = \Re \left[\int_{Q_{m,n}} (L(\zeta) - L(\zeta_{m,n})) dm_\zeta \right].$$

We apply the second order Taylor expansion with the remainder term in the integral form:

$$L(\zeta) - L(\zeta_{m,n}) = L'(\zeta_{m,n})(\zeta - \zeta_{m,n}) + \frac{1}{2} L''(\zeta_{m,n})(\zeta - \zeta_{m,n})^2 + \frac{1}{2} \int_{\zeta_{m,n}}^{\zeta} L'''(s)(\zeta - s)^2 ds.$$

and use the fact that

$$\int_{Q_{m,n}} (\zeta - \zeta_{m,n}) dm_\zeta = \int_{Q_{m,n}} (\zeta - \zeta_{m,n})^2 dm_\zeta = 0.$$

Then

$$\begin{aligned} |j_{m,n}| = \left| \int_{Q_{m,n}} \int_{\zeta_{m,n}}^{\zeta} (\zeta - s)^2 \left(\frac{1}{(s-z)^3} - \frac{1}{s^3} \right) ds dm_\zeta \right| \\ \prec \frac{1}{\text{dist}(z, Q_{m,n})^3} + \frac{1}{\text{dist}(0, Q_{m,n})^3}, \end{aligned}$$

which implies that

$$\mathfrak{S}_1(R, z) = O(1).$$

Finally,

$$\mathfrak{S}_2(R, z) = |W_R(z) - w_R(z)| = \left| b^2 \left(\sum_{\text{dist}(z, A_k) \leq M+10} + \sum_{\text{dist}(z, A_k) > M+10} \right) \underbrace{\int_{A_k} \left(\log \left| 1 - \frac{z}{\zeta_k} \right| - \log \left| 1 - \frac{z}{\zeta} \right| \right) dm_\zeta}_{i_k} \right|,$$

with M as in (4.1). The first term in the right-hand side contains just a finite number of summands and is always bounded. In order to estimate each i_k from the second term we use the Taylor formula (now of the first order) with the same function $L(\zeta) = \log(1 - z/\zeta)$. The choice of ζ_k in (4.1) implies that $\int_{A_k} (\zeta - \zeta_k) dm_\zeta = 0$. Arguing as above, we obtain

$$|i_k| \prec \frac{1}{\text{dist}(z, A_k)^2} + \frac{1}{\text{dist}(0, A_k)^2},$$

whence

$$\mathfrak{S}_2(R, z) = O(1).$$

This completes the proof of Lemma 4.1. \square

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